

AN INEQUALITY

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ABSTRACT. In this paper we prove that the weighted linear combination of products of the k -subsets of an n -set of positive real numbers with weight being the harmonic mean of their reciprocal sets is less than or equal to uniformly weighted sum of products of the k -subsets with weight being the harmonic mean of the whole reciprocal set.

1. INTRODUCTION

There is a version of this inequality for 2-subsets of an n -set which appears in [1] page 327, Problem 4 as a short-listed problem for the Forty Seventh *IMO* 2006 held in Ljubljana, Slovenia. This inequality has an interesting generalization as stated in the Main Theorem 2 below.

2. THE MAIN INEQUALITY

Definition 1. Let $A = \{a_1, a_2, \dots, a_n\}$ be an n -set of positive real numbers. Here we allow the numbers to repeat. i.e. $a_i = a_j$ for some $1 \leq i \neq j \leq n$. Let $S = \{i_1, i_2, \dots, i_k\} \subset \{1, 2, 3, \dots, n\}$ be a k -subset for some $k \leq n$. Let $B_S = \{a_{i_1}, a_{i_2}, a_{i_3}, \dots, a_{i_k}\} \subset A$ be its corresponding set. The reciprocal set denoted by B_S^{-1} of the set B_S is defined to be the set $B_S^{-1} = \{\frac{1}{a_{i_1}}, \frac{1}{a_{i_2}}, \frac{1}{a_{i_3}}, \dots, \frac{1}{a_{i_k}}\}$.

Now we state the main theorem.

Theorem 2. Let $[n] = \{1, 2, 3, \dots, n\}$ denote the set of first n natural numbers. Let $A = \{a_i : i = 1, \dots, n\}$ be an n -set of positive real numbers. For any subset $S \subset [n]$, let $\prod a_S$ denote $\prod_{i \in S} a_i$ and $\sum a_S$ denote $\sum_{i \in S} a_i$. Then

$$(3) \quad \sum_{k\text{-subset } S \subset [n]} \left(\frac{\prod a_S}{\sum a_S} \right) \leq \frac{n}{k} \left(\frac{\sum_{k\text{-subset } S \subset [n]} \prod a_S}{\sum a_{[n]}} \right)$$

i.e. The weighted linear combination of products of the k -sets with weight as harmonic mean of their reciprocal sets is less than or equal to uniformly weighted sum of products of the k -sets with weight the harmonic mean of the whole reciprocal set.

We also observe that the equality occurs if and only if $a_1 = a_2 = \dots = a_n$.

3. SOME SIMPLE CASES OF THE GENERAL INEQUALITY

We prove a few lemmas.

Lemma 4. Let a_1, a_2, a_3 be three positive real numbers. Then the sum of the reciprocals of a_i is greater than or equal to sum of the reciprocals of their pairwise averages.

$$(5) \quad \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \geq \frac{1}{\frac{a_1+a_2}{2}} + \frac{1}{\frac{a_2+a_3}{2}} + \frac{1}{\frac{a_3+a_1}{2}}$$

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Proof. We have from $AM - HM$ inequality applying to the reciprocals $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$ we get

$$\begin{aligned}\frac{\frac{1}{a_1} + \frac{1}{a_2}}{2} &\geq \frac{2}{a_1 + a_2} \\ \frac{\frac{1}{a_2} + \frac{1}{a_3}}{2} &\geq \frac{2}{a_2 + a_3} \\ \frac{\frac{1}{a_3} + \frac{1}{a_1}}{2} &\geq \frac{2}{a_3 + a_1}\end{aligned}$$

and adding these inequalities we have

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \geq \frac{1}{\frac{a_1+a_2}{2}} + \frac{1}{\frac{a_2+a_3}{2}} + \frac{1}{\frac{a_3+a_1}{2}}$$

Hence the Lemma 4 follows. \square

Lemma 6. *Let a_1, a_2, a_3 be three positive real numbers. Then*

$$(7) \quad \frac{a_1 a_2}{a_1 + a_2} + \frac{a_2 a_3}{a_2 + a_3} + \frac{a_3 a_1}{a_1 + a_3} \leq \frac{3(a_1 a_2 + a_2 a_3 + a_3 a_1)}{2(a_1 + a_2 + a_3)}$$

Proof. In order to prove Lemma 6 first we make a simplification by assuming without loss of generality that $a_1 + a_2 + a_3 = 1$. This can be done by normalizing with $a_1 + a_2 + a_3$. Now

$$\begin{aligned}& 2\frac{a_1 a_2}{a_1 + a_2} + 2\frac{a_2 a_3}{a_2 + a_3} + 2\frac{a_3 a_1}{a_1 + a_3} - 2(a_1 a_2 + a_2 a_3 + a_3 a_1) \\ &= a_1 a_2 a_3 \left(\frac{1}{\frac{a_1+a_2}{2}} + \frac{1}{\frac{a_2+a_3}{2}} + \frac{1}{\frac{a_3+a_1}{2}} \right) \\ &\leq a_1 a_2 a_3 \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right) \text{ Using Lemma 4} \\ &= a_1 a_2 + a_2 a_3 + a_3 a_1\end{aligned}$$

Hence the Lemma 6 follows. \square

Lemma 8. *Let $a_1, a_2, a_3, \dots, a_n$ be n positive real numbers. Then the sum of the reciprocals of a_i is greater than or equal to the sum of the reciprocals of their $(n-1)$ -wise averages.*

$$(9) \quad \begin{aligned}\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_n} &\geq \frac{1}{\frac{a_1+a_2+\dots+a_{n-1}+a_n-a_n}{n-1}} + \frac{1}{\frac{a_1+a_2+a_3+\dots+a_n-a_{n-1}}{n-1}} \\ &\quad + \frac{1}{\frac{a_1+a_2+a_3+\dots+a_n-a_{n-2}}{n-1}} + \dots + \frac{1}{\frac{a_1+a_2+a_3+\dots+a_n-a_1}{n-1}}\end{aligned}$$

Proof. This is a generalization of Lemma 4 to n -positive real numbers a_1, a_2, \dots, a_n . We have from $AM - HM$ inequality applying to the reciprocals $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \dots, \frac{1}{a_n}$ we get the following set of inequalities. For every $1 \leq j \leq n$, we get

$$\frac{1}{(n-1)} \left(\sum_{i \neq j, i=1}^n \frac{1}{a_i} \right) \geq \frac{(n-1)}{\sum_{i \neq j, i=1}^n a_i}$$

and adding these inequalities we have

$$\begin{aligned}\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_n} &\geq \frac{1}{\frac{a_1+a_2+\dots+a_{n-1}+a_n-a_n}{n-1}} + \frac{1}{\frac{a_1+a_2+a_3+\dots+a_n-a_{n-1}}{n-1}} \\ &\quad + \frac{1}{\frac{a_1+a_2+a_3+\dots+a_n-a_{n-2}}{n-1}} + \dots + \frac{1}{\frac{a_1+a_2+a_3+\dots+a_n-a_1}{n-1}}\end{aligned}$$

Hence the Lemma 8 follows. \square

Lemma 10. Let $[n] = \{1, 2, 3, \dots, n\}$ denote the set of first n natural numbers. Let $A = \{a_i : i = 1, \dots, n\}$ be a set of n positive real numbers. Then

$$(11) \quad \sum_{(n-1)\text{-subset } S \subset [n]} \left(\frac{\prod a_S}{\sum a_S} \right) \leq \frac{n}{(n-1)} \left(\frac{\sum_{(n-1)\text{-subset } S \subset [n]} \prod a_S}{\sum a_{[n]}} \right)$$

Proof. This is a generalization of the above Lemma 6 to the case of $(n-1)$ -subsets of an n -set. The proof is similar to the proof of Lemma 6 except here we use Lemma 8 instead of Lemma 4. \square

Lemma 12. Let $A = \{a_i : i = 1, \dots, n\}$ be a set of n -positive real numbers. Then

$$(13) \quad \sum_{i < j} \frac{a_i a_j}{a_i + a_j} \leq \frac{n}{2(a_1 + a_2 + \dots + a_n)} \sum_{i < j} a_i a_j$$

Proof. The proof is as follows. Again by normalizing with $\sum_{i=1}^n a_i$ we can assume that $\sum_{i=1}^n a_i = 1$ and it is enough to prove that

$$(14) \quad \sum_{i < j} \frac{a_i a_j}{a_i + a_j} \leq \frac{n}{2} \sum_{i < j} a_i a_j$$

So consider

$$\begin{aligned} & 2 \sum_{i < j} \frac{a_i a_j}{a_i + a_j} - 2 \sum_{i < j} a_i a_j = \sum_{i < j} \frac{2}{a_i + a_j} \left(\sum_{k \neq i, k \neq j} a_i a_j a_k \right) \\ & = \left(\sum_{3\text{-subset } S \subset [n]} \left(\sum_{2\text{-subset } T \subset S} \frac{2 \prod a_S}{\sum a_T} \right) \right) \\ & \text{Now using Lemma 4 for all 3 subsets of } \{1, 2, 3, \dots, n\} \text{ we get} \\ & \leq (n-2) \left(\sum_{2\text{-subset } S \subset [n]} \prod a_S \right) \end{aligned}$$

Hence the lemma follows. \square

4. PROOF OF THE MAIN THEOREM

Here we prove the Main Theorem 2

Proof. Now we generalize to the case given in the Theorem 2 by first normalizing the inequality with $\sum a_i$ so that we can assume that $\sum a_i = 1$. And we have to proof the following inequality

$$\sum_{k\text{-subset } S \subset [n]} \left(\frac{\prod a_S}{\sum_k a_S} \right) \leq n \left(\sum_{k\text{-subset } S \subset [n]} \prod a_S \right)$$

We have

$$\begin{aligned}
& \sum_{k\text{-subset } S \subset [n]} \left(k \frac{\prod a_S}{\sum a_S} \right) - k \left(\sum_{k\text{-subset } S \subset [n]} \prod a_S \right) \\
&= k \left(\sum_{k\text{-subset } S \subset [n]} \frac{\prod a_S (1 - \sum a_S)}{\sum a_S} \right) \\
&= k \left(\sum_{(k+1)\text{-subset } S \subset [n]} \left(\sum_{k\text{-subset } T \subset S} \frac{\prod a_S}{\sum a_T} \right) \right) \\
&\leq (n - k) \left(\sum_{k\text{-subset } S \subset [n]} \prod a_S \right) \text{ Using inequality 9 in Lemma 8}
\end{aligned}$$

The equality occurs when if and only all the AM-HM inequalities involved give equality which holds if and only if $a_1 = a_2 = \dots = a_n$. Hence the Main Theorem 2 follows. \square

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